Mathematical aspects of the cold plasma model

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Abstract

A simple model for electromagnetic wave propagation through zero-temperature plasma is analyzed. Many of the complexities of the plasma state are present even under these idealized conditions, and a number of mathematical difficulties emerge. In particular, boundary value problems formulated on the basis of conventional electromagnetic theory turn out to be ill-posed in this context. However, conditions may be prescribed under which solutions to the Dirichlet problem exist in an appropriately weak sense. In addition to its physical interest, analysis of the cold plasma model illuminates generic difficulties in formulating and solving boundary value problems for mixed elliptic-hyperbolic partial differential equations.

1 Introduction

Among the many equations of mathematical physics which change from elliptic to hyperbolic type along a smooth curve, only the equations for transonic flow have received sustained attention. In this brief review we consider elliptic-hyperbolic equations originating in a simple model for the propagation of electromagnetic waves through zero-temperature plasma. Solutions to such equations are likely to have significantly weaker regularity than solutions to the linearized equations of transonic flow. Recognizing the interdisciplinary nature of the topic, we assume a familiarity with physics but not necessarily plasma physics, and analysis but not necessarily elliptic-hyperbolic equations. However, the physics is confined to a review of fundamental results in Sec. 2, whereas the mathematical results of Sec. 3 are somewhat more technical. There we consider the extent to which problems formulated primarily for linearized equations of gas dynamics possess analogies for equations arising from a different physical problem. Continuing such investigations in various physical and geometric contexts (*c.f.* [29]), one may hope to obtain eventually a natural theory for linear elliptic-hyperbolic partial differential equations.

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1.1 Physical background

The plasma state is characterized by the dominance of long-range, nonlinear effects. For matter in such a state, it is particularly difficult to obtain mathematical problems which can be stated with a satisfactory degree of rigor, and for which solutions can be shown to exist. Without a proof of the existence and uniqueness of solutions — which, in particular, specifies the function spaces in which solutions lie — it is hard to place appropriate boundary conditions on numerical experiments and to gauge the reliability of the results obtained.

If one hopes to obtain a tractable mathematical problem, it is usually necessary to impose harsh assumptions on both the plasma and the applied field. Perhaps the harshest of these fixes the temperature of the plasma to be zero. This permits one to neglect altogether the fluid properties of the medium, which is then treated as a linear dielectric. Somewhat surprisingly, the assumption of zero plasma temperature is a useful first approximation to the products of tokamaks: low-density plasmas which are remarkably free of expected high-temperature phenomena such as collisions and wall effects. See the remarks in the introduction to [36] and the more detailed discussions in [39]. More generally, the cold plasma model approximates the effects of small-amplitude electromagnetic waves, propagating with phase velocities which are sufficiently large in comparison to the thermal velocity of the particles.

We note that the term *cold plasma* is highly ambiguous. Although we take this to imply zero temperature, in the astrophysics literature interstellar plasmas on the order of 10^4 K to 10^5 K are typically referred to as "cold" (see, *e.g.*, [11]). Very recently, "ultracold" neutral plasmas, having electron temperatures in the range from 1 K to 10^3 K and ion temperatures ranging from 10^{-3} K to 1 K, have been created experimentally. The cold plasma model explored in this paper is apparently too simple to yield quantitative insight into those plasmas. In particular, the fluid dynamics aspects of experimental ultracold plasmas appear to be non-negligible (*c.f.* Sec. 3 of [17]).

The other physical hypotheses imposed in this review are also quite restrictive: Although the plasma is not assumed to be homogeneous, the inhomogeneity is taken to be two-dimensional, so the governing equations for the model are also essentially two-dimensional. Moreover, the applied magnetic field is assumed in Sec. 2.4 to be longitudinal and the resulting wave motion confined to electrostatic oscillations. In Sec. 2.5 we consider electromagnetic waves, but we find (after reviewing a detailed analysis by Weitzner [39]) that elliptic-hyperbolic equations arising in the electrostatic case retain their validity as a qualitative model for the general case.

For the most part, the outstanding mathematical problems relevant to the cold plasma model are boundary value problems for Maxwell's equations. The dielectric tensor for these equations will render them of elliptic type on one part of their domain and of hyperbolic type on the remainder, except for a smooth curve (the *parabolic line*) separating the two regions. Little is known about the formulation of well-posed boundary value problems for equations which change type in this way, especially as the equations that arise in the cold plasma model

appear to have certain fundamental differences from those that arise in gas dynamics.

Careful reasoning about the mathematical properties of plasma models is not needed merely in order to prevent "mathematicians' nightmares." An example is known [26] in which the boundary conditions suggested by physical reasoning about the plasma lead to a mathematically ill-posed problem in the expected function space. Moreover, numerical experiments tend to confirm the difficulties that arise when the model equations are subjected to classical analytic techniques; see [26] and various remarks in [39].

High-frequency waves can be modelled via geometrical optics. (Any propagating electromagnetic field will tend to have high frequency relative to the characteristic plasma frequencies; see, *e.g.*, Sec. 2.4 of [23].) Mathematical problems that arise in the geometrical optics approximation are quite different from those that arise from applying Maxwell's equations directly, and we do not pursue the geometrical optics approach in this review. The complexity of the geometrical optics approximation is due to significant difference in magnitude among the terms of the plasma conductivity tensor at lower hybrid frequencies; see [34] and the references therein.

The physics presented in Secs. 2.1 and 2.2 essentially goes back to the work of Tonks and Langmuir [38] in the late 1920s. The results of Sec. 2.3 were already well known in the 1950s [2, 3, 35]; those of Secs. 2.4 and 2.5 date from the 1970s [20, 33] and 1980s, respectively. In particular, Sec. 2.5 derives some fundamental analytic formulas introduced by Weitzner in [39] and [40]; see also [16]. Section 3 is based on recent results, [27, 28] which extend analogous research on equations of Tricomi type — particularly [21] and [22]; see also [41], an earlier paper which is based on [24].

In the sequel, a subscripted variable denotes (usually partial) differentiation in the direction of the variable, whereas subscripted numbers denote components of a matrix, vector, or tensor. Differentiation of vector or matrix components in the direction of a variable is indicated by preceding the subscripted variable by a comma. Unless otherwise stated, a cartesian coordinate system is assumed in which the subscript 1 denotes a component projected onto the x-axis, the subscript 2 denotes a component projected onto the y-axis, and the subscript 3 denotes a component projected onto the z-axis. In particular, $\mathbf{x} = (x_1, x_2, x_3) = (x, y, z)$ and we denote the canonical cartesian basis by $\left(\hat{\imath}, \hat{\jmath}, \hat{k}\right)$.

2 The cold plasma model

A *plasma* is a fluid composed of electrons and one or more species of ions. Because it is a fluid, its evolution must satisfy the equations of fluid dynamics. But because the particles of the fluid are charged, they act as sources of an electromagnetic field, which is governed by Maxwell's equations. The presence of this intrinsic field leads to highly nonlinear behavior. Indeed, the dominance of long-range electromagnetic interactions over the short-range interatomic or intermolecular forces is often cited as the defining characteristic of the plasma

state.

If the plasma is at zero temperature, then Amontons' Law implies that the pressure term in the equations for fluid motion will also be zero, and the laws of fluid dynamics will enter only through the conservation laws for mass and momentum. In fact, because collisions can be neglected, the fluid aspect of the medium can be virtually ignored. The plasma is then represented as a static dielectric through which electromagnetic waves propagate.

In particular, *zero-order quantities* — the plasma density, proportions of ions to electrons, and the background magnetic field — can all be considered static in time and uniform in space. *First-order quantities* — the electric field $\mathbf{E}$ and particle velocities $\mathbf{v}$, are assumed to be expressible as plane waves: sinusoidal waves proportional to functions having the form $\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right]$, where $\mathbf{k}$ is the propagation vector of the wave (not to be confused with the notation for the cartesian basis vector $\hat{k}$); $\mathbf{r}$ is the radial coordinate in space; ω is angular frequency; $i^2=-1$. Thus in cartesian coordinates, $\mathbf{k}\cdot\mathbf{r}=k_1x+k_2y+k_3z$.

In the following we review basic elements of the physical theory that results from these assumptions. The material in Secs. 2.1–2.3 is completely standard and can be found in many sources. The classical reference is Ch. 1 of [36]; see also [1], [8], [13], and Sec. 2 of [40]. More recent surveys include nodes 43–45 of [12] and Ch. 2 of [37]. A recent review of theoretical plasma physics can be found in [6]. We employ *SI* units except where other units are specified.

2.1 Equations of motion

Consider a single particle of mass m, having charge $q=Z\delta e$, where Z is a positive integer, δ equals 1 or -1, and e is the charge on an electron. Let the particle be subjected only to the Lorentz force

$$\mathbf{F}_L=q\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right),$$

where

$$\mathbf{B}=B_0\hat{k}. \tag{2.1}$$

Equation (2.1) implies that the applied magnetic field is *longitudinal*: its only nonzero component is directed along the positive z-axis. (In fact, there is little harm in assuming, somewhat more generally, that

$$\mathbf{B}=B_0\hat{k}+\tilde{\mathbf{B}}\left(x,y,z\right)\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right],$$

with $|\tilde{\mathbf{B}}|<<|B_0|$.)

The equation of motion for the particle is given by Newton's Second Law of Motion, that is,

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}_L. \tag{2.2}$$

In accordance with our assumption about first-order quantities, we write

$$\mathbf{v}\left(x,y,z,t\right)=\tilde{\mathbf{v}}\left(x,y,z\right)\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right],$$

or

$$\frac{d\mathbf{v}}{dt} = -i\omega\mathbf{v}.$$

Substituting this result into (2.2) yields

$$-im\omega\tilde{\mathbf{v}} = q\left(\tilde{\mathbf{E}} + \tilde{\mathbf{v}} \times \mathbf{B}\right), \tag{2.3}$$

where

$$\mathbf{E}(x,y,z,t) = \tilde{\mathbf{E}}(x,y,z)\exp\left[i\left(\mathbf{k}\cdot\mathbf{r} - \omega t\right)\right]. \tag{2.4}$$

Initially we will take $\tilde{\mathbf{E}}$ to be a constant vector:

$$\tilde{\mathbf{E}}(x,y,z) = E_1\hat{\imath} + E_2\hat{\jmath} + E_3\hat{k}, \tag{2.5}$$

where E_1, E_2, and E_3 are constants, and similarly for $\tilde{\mathbf{v}}$.

Defining the *cyclotron frequency*

$$\Omega = \left|\frac{qB_0}{m}\right|,$$

Eq. (2.3) has solutions $\mathbf{v} = (v_1, v_2, v_3)$ satisfying

$$v_1 = \frac{iq}{m\left(\omega^2 - \Omega^2\right)}\left(\omega E_1 + i\delta\Omega E_2\right); \tag{2.6}$$

$$v_2 = \frac{iq}{m\left(\omega^2 - \Omega^2\right)}\left(\omega E_2 - i\delta\Omega E_1\right); \tag{2.7}$$

$$v_3 = \frac{iq}{m\omega}E_3. \tag{2.8}$$

2.2 The dielectric tensor

Although the above relations were derived for an individual particle, they also hold, in our simplified linear model, for each species of particle in a plasma consisting of electrons and $N-1$ species of ions. In particular, the plasma current can be written as the sum

$$\mathbf{j} = \sum_{\nu=1}^{N} n_\nu q_\nu \mathbf{v}_\nu, \tag{2.9}$$

where n_ν is the density of particles having charge magnitude $|q_\nu| = Z_\nu e$.

In the sequel we will only consider the aggregate of particles, in which Eqs. (2.1)–(2.8) pertain with the quantities $\mathbf{v}$, m, q, Z, δ, and Ω indexed by ν, where $\nu = 1, ..., N$. Introduce the *electric displacement vector*

$$\mathbf{D} = \text{vacuum displacement} + \text{plasma current} = \epsilon_0\mathbf{E} + \frac{i}{\omega}\mathbf{j}, \tag{2.10}$$

where ϵ_0 is the permittivity of free space. It will be convenient to express (2.10) in the form

$$\mathbf{D} = \epsilon_0\mathbf{K}\mathbf{E}, \tag{2.11}$$

where

$$D_i = \epsilon_0 \sum_{j=1}^{3} K_{ij} E_j \tag{2.12}$$

and $\mathbf{K} = (K_{ij})$ is the *dielectric tensor* (also called the *cold plasma conductivity tensor*). The tensorial nature of this quantity reflects the anisotropy of the plasma due to the presence of an applied magnetic field. (Note that in the sequel the reader will be expected to distinguish between the notation $\mathbf{K}$ for the dielectric tensor, the notation K_{ij} for the scalar element of its i^{th} row and j^{th} column, and the notation $\mathcal{K}$ for the type-change function of an elliptic-hyperbolic equation.)

Equations (2.6)–(2.11) imply that

$$\mathbf{K} = \begin{pmatrix} s & -id & 0 \\ id & s & 0 \\ 0 & 0 & p \end{pmatrix}, \tag{2.13}$$

where s, d, and p are defined in terms of:

i) the *plasma frequency*, which for particles of the ν^{th} species is given by

$$\Pi_\nu^2 = \frac{n_\nu q^2}{\epsilon_0 m_\nu};$$

ii) the *permittivities* R or L of a right- or left-circularly polarized wave travelling in the direction $\hat{k}$; these are given by

$$R = 1 - \sum_{\nu=1}^{N} \frac{\Pi_\nu^2}{\omega^2} \left(\frac{\omega}{\omega + \delta_\nu \Omega_\nu} \right)$$

and

$$L = 1 - \sum_{\nu=1}^{N} \frac{\Pi_\nu^2}{\omega^2} \left(\frac{\omega}{\omega - \delta_\nu \Omega_\nu} \right).$$

In terms of these quantities,

$$s = \frac{1}{2} (R + L),$$

$$d = \frac{1}{2} (R - L),$$

and

$$p = 1 - \sum_{\nu=1}^{N} \frac{\Pi_\nu^2}{\omega^2}.$$

The mass of an electron is considerably smaller than the mass of any ion; so the squared ion cyclotron frequencies obtained from combining fractions in R and L can be neglected, leading to the approximate formulas

$$R \approx 1 - \sum_{\nu=1}^{N-1} \frac{\Pi_e^2}{\omega^2 + \omega\Omega_e + \Omega_e \Omega_{i_\nu}} \tag{2.14}$$

and

$$L \approx 1 - \sum_{\nu=1}^{N-1} \frac{\Pi_e^2}{\omega^2 - \omega\Omega_e + \Omega_e\Omega_{i_\nu}}. \tag{2.15}$$

In these formulas, the subscripted e denotes the value of the relevant quantity for the electrons and the subscripted i_ν denotes that value for the ν^{th} species of ion. Note that, by the same reasoning, the ion plasma frequencies can be neglected in the definition of p.

2.3 The plasma dispersion relation

The field equations for the system described in Sec. 2.2 are Maxwell's equations,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.16}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right), \tag{2.17}$$

where μ_0 is the permeability of free space.

From the form of Eq. (2.4), it is clear that whenever $\mathbf{E}$ and $\mathbf{B}$ are plane waves, Eqs. (2.16) and (2.17) reduce to the simpler form

$$\mathbf{k} \times \mathbf{E} = \omega \mathbf{B} \tag{2.18}$$

and

$$\mathbf{k} \times \mathbf{B} = -i\mu_0 \mathbf{j} - \omega\mu_0\epsilon_0 \mathbf{E}. \tag{2.19}$$

We can rewrite Eq. (2.19) to read

$$\begin{aligned} \mathbf{k} \times \mathbf{B} &= -\omega\mu_0 \left(\frac{i\mathbf{j}}{\omega} + \epsilon_0 \mathbf{E} \right) \\ &= -\omega\mu_0 \mathbf{D} = -\epsilon_0\mu_0\omega \mathbf{K}\mathbf{E}, \end{aligned} \tag{2.20}$$

where we have used (2.10) and (2.11) in deriving the last identity.

Now using (2.18), (2.20), and the elementary identity $\mu_0\epsilon_0 = c^{-2}$, where c is the speed of light *in vacuo*, we obtain

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) = \mathbf{k} \times (\omega \mathbf{B}) = \omega (\mathbf{k} \times \mathbf{B})$$

$$= -\omega^2 \epsilon_0 \mu_0 \mathbf{K}\mathbf{E} = -\left(\frac{\omega}{c}\right)^2 \mathbf{K}\mathbf{E},$$

implying that

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) + \left(\frac{\omega}{c}\right)^2 \mathbf{K}\mathbf{E} = 0. \tag{2.21}$$

Define the *index of refraction vector*

$$\mathbf{n} = \frac{c}{\omega}\mathbf{k}.$$

With this construction, the scalar index of refraction acquires a direction: that of the wave propagation vector $\mathbf{k}$. In terms of $\mathbf{n}$, Eq. (2.21) reads

$$\mathbf{n} \times (\mathbf{n} \times \mathbf{E}) + \mathbf{K}\mathbf{E} = 0. \tag{2.22}$$

Conventionally, $\mathbf{k}$ (and thus $\mathbf{n}$) lies in the xz-plane. Denote by θ the angle subtended by the vectors $\mathbf{k}$ and $\mathbf{B}$. Then (2.22) can be written as the matrix equation

$$\begin{pmatrix} s - n^2\cos^2\theta & -id & n^2\cos\theta\sin\theta \\ id & s - n^2 & 0 \\ n^2\cos\theta\sin\theta & 0 & p - n^2\sin^2\theta \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = 0.$$

This matrix equation has a nontrivial solution precisely when the determinant of the 3×3 matrix vanishes. The condition for the vanishing of that determinant, the *cold plasma dispersion relation* is, geometrically, the equation for the wave-normal surface:

$$An^4 - Bn^2 + C = 0, \tag{2.23}$$

where the coefficients satisfy

$$A = s\sin^2\theta + p\cos^2\theta, \tag{2.24}$$

$$B = \left(s^2 - d^2\right)\sin^2\theta + ps\left(1 + \cos^2\theta\right), \tag{2.25}$$

and

$$C = p\left(s^2 - d^2\right). \tag{2.26}$$

Because the left-hand side of Eq. (2.23) is a quadratic polynomial in n^2, we obtain from the quadratic formula the solutions

$$n^2 = \frac{B \pm F}{2A}$$

for F satisfying $F^2 = B^2 - 4AC$. Using (2.24)–(2.26) to write

$$F^2 = (RL - ps)^2\sin^4\theta + 4p^2d^2\cos^2\theta,$$

we obtain

$$\tan^2\theta = -\frac{p\left(n^2 - R\right)\left(n^2 - L\right)}{\left(sn^2 - RL\right)\left(n^2 - p\right)}.$$

These equations yield criteria for *cutoff*, where $n = 0$, or *resonance*, where $n \to \infty$.

Physically, cutoffs and resonances correspond to a change in the behavior of the wave from possible propagation to evanescence. Mathematically, we will identify certain resonances with a change in type of the governing field equation, from hyperbolic (implying wave propagation) to elliptic (implying evanescence). These transitions may be accompanied, under certain conditions, by reflection and/or absorption of the wave.

Sufficient conditions for cutoff are $p = 0$, $R = 0$, or $L = 0$ — that is, a sufficient condition is $C = 0$. A sufficient condition for resonance is $A = 0$ which, given Eq. (2.24), can be written

$$\tan^2 \theta = -\frac{p}{s}. \tag{2.27}$$

Particular cases of interest are $\theta = 0$ (propagation parallel to the magnetic field) and $\theta = \pi/2$ (propagation perpendicular to the magnetic field). We will be particularly interested in the *hybrid resonances* at $\theta = \pi/2$, which occur at frequencies for which $s = 0$.

2.4 Electrostatic waves

The electric field is said to be *electrostatic* if it approximately satisfies

$$\mathbf{E} = -\nabla\Phi, \tag{2.28}$$

where Φ is a scalar potential. Equation (2.28) is satisfied locally by all time-independent electric fields and in an ordinary dielectric, the converse is also true. However in cold plasma there also exist time-dependent solutions of (2.28). Cold plasma has been characterized as a linear dielectric through which electromagnetic waves propagate. Thus these waves include, in distinction to ordinary dielectrics, the special case of propagating electrostatic waves.

We write Φ in the form

$$\Phi(x, y, z; t) = \varphi(x, y, z) \exp\left[\mathbf{k} \cdot \mathbf{r} - i\omega t\right],$$

and add to Maxwell's equations (2.16), (2.17) the additional equation

$$div\, \mathbf{D} = 0, \tag{2.29}$$

which follows from Gauss' law for electricity.

Equation (2.28) implies immediately that

$$\nabla \times \mathbf{E} = 0. \tag{2.30}$$

This is most easily seen if we use differential forms, and note that in terms of the exterior derivative, $\mathbf{E} = d\Phi$, so (2.30) is just the well-known property that

$$d\mathbf{E} = d^2\Phi = 0.$$

(Here and below we will switch from vectors to forms whenever the calculation is made more transparent thereby; but we will not change notation for the underlying geometric object, making the convention that the argument of the exterior derivative is always assumed to be a differential form.) In either the vector or form notation, identity (2.30) follows from the equality of mixed partial derivatives. Applying the arguments relating (2.16) to (2.18) and (2.17) to (2.19) (*translation into Fourier modes*), we rewrite (2.30) in the form

$$\mathbf{k} \times \mathbf{E} = 0.$$

This implies, by the properties of the cross product, the geometric fact that the vectors $\mathbf{k}$ and $\mathbf{E}$ are parallel. We say that electrostatic waves are *longitudinal*. Physically, they appear as oscillations along the axis of the magnetic field.

Thus we conclude that *transverse* waves, which propagate in a direction perpendicular to the magnetic field, must satisfy

$$\mathbf{k} \cdot \mathbf{E} = 0.$$

Again, differential forms are illuminating: The above identity becomes $\delta d\Phi = 0$, where δ is the formal adjoint of the exterior derivative d. But Φ is a 0-form, so $\delta\Phi = 0$ automatically, and we find that transverse waves satisfy

$$\delta d\Phi + d\delta\Phi \equiv \Delta\Phi = 0,$$

that is, transverse waves are necessarily harmonic.

2.4.1 Plane-layered media

If we allow a plane-layered inhomogeneous medium (parameterized by x), the electrostatic potential has the form

$$\Phi\left(x, y, z\right) = \varphi(x) \exp\left[i\left(k_2 y + k_3 z\right)\right],$$

where k_j is the j^{th} component of the wave vector $\mathbf{k}$ for $j = 1, 2, 3$. Substitution of this form for the electric potential into Eq. (2.29) yields, using (2.12), the single scalar equation [20]

$$K_{11}\varphi_{xx} + \left(K_{11,x} + i\sigma_0\right)\varphi_x = 0, \tag{2.31}$$

where

$$\sigma_0 = k_3\left(K_{13} + K_{31}\right) + k_2\left(K_{12} + K_{21}\right),$$

and zero-order terms in φ have been neglected. This equation has a power-series solution except where K_{11} vanishes.

Explicit solutions of the model equation (2.31) under various physical assumptions are given in Sec. 1 of [33], the Appendix to [20], and Sec. C of [16].

It is easy to believe that inhomogeneities may develop in a plasma. For example, if the temperature is not exactly zero, the difference in velocity between electrons and ions can be expected to destabilize an initially homogeneous distribution. However, it is difficult to imagine a force that will restrict these inhomogeneities to a 1-parameter foliation, which would be necessary in order to arrive at Eq. (2.31). Formally, an electromagnetic potential leading to Eq. (2.31) could be induced by applying a driving potential to the metallic plates of a condenser. But in practice, this plasma geometry has little application either in the laboratory or in nature.

2.4.2 A two-dimensional inhomogeneity

Suppose instead that the medium is a cold, anisotropic plasma with a two-dimensional inhomogeneity parameterized by two variables, x and z. Then the field potential has the form

$$\Phi(x, y, z) = \varphi(x, z) \exp[ik_2 y].$$

The electric field $\mathbf{E}$ is then given by

$$E = -\nabla\Phi = (E_1, E_2, E_3) = -\left(\varphi_x e^{ik_2 y}, ik_2 \varphi e^{ik_2 y}, \varphi_z e^{ik_2 y}\right).$$

Maxwell's equations for the electric displacement vector $\mathbf{D} = (D_1, D_2, D_3)$ take the form

$$0 = \nabla \cdot \mathbf{D} = D_{1,x} + D_{2,y} + D_{3,z}. \tag{2.32}$$

We continue to neglect those terms which do not contain derivatives of φ, as φ is assumed to oscillate rapidly.

Because neither φ nor K_{ij} have any dependence on y, the problem is effectively two-dimensional. Applying Eq. (2.12), the surviving terms of Eq. (2.32) are (setting ϵ_0 equal to 1)

$$D_{1,x} = -\left[K_{11}\varphi_{xx} + K_{11,x}\varphi_x + K_{12}\varphi_x ik_2 + K_{13}\varphi_{zx} + K_{13,x}\varphi_z\right] e^{ik_2 y};$$

$$D_{2,y} = -\left[K_{21}\varphi_x ik_2 + K_{23}\varphi_z ik_2\right] e^{ik_2 y};$$

$$D_{3,z} = -\left[K_{31}\varphi_{xz} + K_{31,z}\varphi_x + K_{32} ik_2 \varphi_z + K_{33}\varphi_{zz} + K_{33,z}\varphi_z\right] e^{ik_2 y}.$$

Collecting terms, we find that [33]

$$K_{11}\varphi_{xx} + 2\sigma\varphi_{xz} + K_{33}\varphi_{zz} + \alpha_1\varphi_x + \alpha_2\varphi_z = 0, \tag{2.33}$$

where

$$2\sigma = K_{13} + K_{31};$$

$$\alpha_1 = K_{11,x} + ik_2(K_{12} + K_{21}) + K_{31,z};$$

$$\alpha_2 = K_{13,x} + ik_2(K_{23} + K_{32}) + K_{33,z}.$$

Two-dimensional inhomogeneities of the kind represented by Eq. (2.33) can be expected to arise in toroidal fields, such as those created in tokamaks.

The entries of the matrix K under our assumptions on $\mathbf{B}_0$ imply that $\sigma = 0$, so we can write Eq. (2.33) in the form

$$K_{11}\varphi_{xx} + K_{33}\varphi_{zz} + \text{lower-order terms} = 0. \tag{2.34}$$

Equation (2.34) is of either elliptic or hyperbolic type, depending on whether the sign of the product

$$K_{11} \cdot K_{33} = \left(1 - \sum_{\nu=1}^{N} \frac{\Pi_\nu^2}{\omega^2 - \Omega_\nu^2}\right) \cdot \left(1 - \sum_{\nu=1}^{N} \frac{\Pi_\nu^2}{\omega^2}\right) \tag{2.35}$$

is, respectively, positive or negative.

The sign of K_{11} changes at the *cyclotron* resonances $\omega^2 = \Omega_\nu^2$. The cold plasma model breaks down at these resonances, as three terms of the dielectric tensor become infinite. The sign of K_{11} also changes at the *hybrid* resonances, at which

$$1 = \sum_{\nu=1}^{N} \frac{\Pi_\nu^2}{\omega^2 - \Omega_\nu^2}. \tag{2.36}$$

(These resonances, which have both a low-frequency and a high-frequency solution, are hybrid in that they involve both plasma and cyclotron frequencies.) In particular, the sign changes at the *lower* hybrid resonance,

$$1 + \frac{\Pi_e^2}{\Omega_e^2} = \frac{\Pi_i^2}{\omega^2}, \tag{2.37}$$

where as before, the subscript e denotes electron frequency, and the subscript i denotes ion frequency. At the hybrid resonance frequencies the cold plasma model retains its validity.

The sign of K_{33} changes on the surface

$$1 = \sum_{\nu=1}^{N} \frac{\Pi_\nu^2}{\omega^2}, \tag{2.38}$$

the resonance at which the frequency of the applied wave equals the plasma frequency of the medium. We may suppose that an electromagnetic wave propagating through a plasma does so at a much higher frequency than any of the characteristic frequencies of the plasma. Otherwise, the plasma magnetic field would prevent the waves from propagating very far (*c.f.* [18]). Thus in evaluating (2.35) and in the sequel we will take K_{33} to be strictly positive.

Borrowing the terminology of fluid dynamics, we will refer to resonances such as (2.36)–(2.38) as *sonic conditions* on Eq. (2.34).

2.4.3 The type of the governing equation

In order to understand the possible variants of Eq. (2.34), we consider the coordinate transformation $(x, z) \to (\xi(x, z), \eta(x, z))$, where

$$\xi = K_{11}(x, z).$$

In these coordinates, the higher-order terms of Eq. (2.34) assume the form

$$K_{11}\varphi_{xx} + K_{33}\varphi_{zz} = \left(\xi\xi_x^2 + K_{33}\xi_z^2\right)\varphi_{\xi\xi} + \\ \left(\xi\xi_x\eta_x + K_{33}\xi_z\eta_z\right)\varphi_{\xi\eta} + \left(\xi\eta_x^2 + K_{33}\eta_z^2\right)\varphi_{\eta\eta}. \tag{2.39}$$

In order that the transformation $(x, z) \to (\xi, \eta)$ be nonsingular, we require that its Jacobian be nonvanishing, *i.e.,*

$$\xi_x\eta_z - \xi_z\eta_x \neq 0. \tag{2.40}$$

Because we want the coefficients of the cross term $\varphi_{\xi\eta}$ to be zero in the new coordinates, we impose the condition that

$$\xi\xi_x\eta_x + K_{33}\xi_z\eta_z = 0. \tag{2.41}$$

Both ξ and K_{33} are given, and it is easy for the two first derivatives of η to satisfy (2.40) and (2.41) simultaneously.

Two possibilities arise. Either

i) ξ and ξ_z never vanish simultaneously, or

ii) there exist one or more points (x, z) on the domain at which

$$\xi\,(x,z) = \xi_z\,(x,z) = 0. \tag{2.42}$$

In case *i)*, the condition $\xi = 0$ implies, via (2.41) and the assumption that K_{33} is positive, the accompanying condition $\eta_z = 0$. But if ξ and η_z both vanish, then the coefficient of $\varphi_{\eta\eta}$ in (2.39) vanishes; that is,

$$\xi\eta_x^2 + K_{33}\eta_z^2 = 0$$

whenever $\xi = 0$. Again using (2.41), we obtain from Eqs. (2.34) and (2.39) an equation with higher-order terms having the form

$$\varphi_{\xi\xi} + \frac{\xi\eta_x^2 + K_{33}\eta_z^2}{\xi\xi_x^2 + K_{33}\xi_z^2}\varphi_{\eta\eta} = 0. \tag{2.43}$$

The denominator in the coefficient of $\varphi_{\eta\eta}$ cannot be zero: ξ and ξ_z cannot vanish simultaneously, and if ξ_x vanishes, then ξ_z must be nonzero in order to preserve condition (2.40). So Eq. (2.43) is of the form

$$\varphi_{\xi\xi} + \mathcal{K}\,(\xi,\eta)\,\varphi_{\eta\eta} = 0, \tag{2.44}$$

where $\mathcal{K}\,(\xi,\eta) = 0$ if and only if $\xi = 0$, an equation of *Tricomi type*.

In case *ii)*, condition (2.40) prevents η_z from vanishing when ξ_z vanishes. Thus if ξ and ξ_z vanish together, the coefficient of $\varphi_{\eta\eta}$ in (2.39) will not vanish at that point. Thus in case *ii)* we obtain from (2.34), (2.39), and (2.41) an equation with higher-order terms having the form

$$\frac{\xi\xi_x^2 + K_{33}\xi_z^2}{\xi\eta_x^2 + K_{33}\eta_z^2}\varphi_{\xi\xi} + \varphi_{\eta\eta} = 0, \tag{2.45}$$

where the numerator in the coefficient of $\varphi_{\xi\xi}$, but not the denominator, is zero whenever ξ is zero. That is, Eq. (2.45) is an equation of the form

$$\mathcal{K}\,(\xi,\eta)\,\varphi_{\xi\xi} + \varphi_{\eta\eta} = 0, \tag{2.46}$$

where $\mathcal{K}\,(\xi,\eta) = 0$ if and only if $\xi = 0$, an equation of *Keldysh type*.

See Sec. 1.2 of [5], [9], Sec. 1 of [26], and Eqs. (75)–(78) of [39] for arguments of this kind.

The regularity of weak solutions to equations of Tricomi type can be established by microlocal arguments; see [14] and [15] and, especially, [30] and [31]. These arguments appear to fail for equations of Keldysh type, and one expects weaker regularity for weak solutions to such equations.

The crucial question is: does condition (2.42) occur in our physical model? The answer to that question is "yes."

2.4.4 Geometry of the resonance curve (*after Piliya and Fedorov*)

Returning to our original xz-coordinates, we set the elements K_{11} and K_{22} of the dielectric tensor equal to $\mathcal{K}$, and the element K_{33} equal to η. The coefficients of the only other nonzero elements, K_{12} and K_{21}, are zero in Eq. (2.34), so only K_{11} and K_{33} play a direct role in the analysis. The sonic condition is equivalent to the alternative:

$$\mathcal{K} = 0 \tag{2.47}$$

or

$$\mathcal{K} \sin^2 \theta + \eta \cos^2 \theta = 0, \tag{2.48}$$

where $\theta\,(x, z)$ is the angle between the direction of $\mathbf{B_0}$ and the xz-plane; *c.f.* (2.27).

The singular points on the *sonic line* (2.47) are the points at which this curve (which is not a generally a line in standard coordinates) is tangent to the projection of the force lines of $\mathbf{B_0}$ in the xz-plane — that is, the flux lines of the magnetic field. The singular points of the graph Γ of Eq. (2.48) are the points at which the flux lines of $\mathbf{B_0}$ are normal to Γ.

This motivates the placement of the origin at a singular point of the sonic line, with the z-axis (the axis along which $\mathbf{B_0}$ is directed) tangent to the sonic line. The x-axis is directed along the inward normal to the sonic line, relative to the hyperbolic region of Eq. (2.33). Then K_{11} and σ both vanish at the origin. Taking both x and z to be small, one can write

$$K_{11} = \frac{x}{a} + \frac{z^2}{b} \tag{2.49}$$

and

$$K_{33} = -\eta_0, \tag{2.50}$$

where η_0 is a positive constant. Scale x and z, via

$$x \to \tilde{x} = x/a \tag{2.51}$$

and

$$z \to \tilde{z} = z/a\sqrt{\eta_0}, \tag{2.52}$$

in order to obtain dimensionless variables $\tilde{x}$ and $\tilde{z}$. In this way, one obtains in place of (2.33) the equation

$$-\left(\tilde{x} + A\tilde{z}^2\right) \varphi_{\tilde{x}\tilde{x}} + \varphi_{\tilde{z}\tilde{z}} - \varphi_{\tilde{x}} = 0, \tag{2.53}$$

where A is a constant, for the simple case in which the coefficients of cross terms vanish identically [33].

2.5 Analytic difficulties in the electromagnetic case (*after H. Weitzner)*

In this section we suppose that the electric field satisfies Eqs. (2.4) and (2.5), but no longer assume that the electric field satisfies condition (2.28). Closely following [39], we attempt to study the resulting field equations using conventional analytic tools, in order to see what difficulties arise.

Repeating the calculations of Eqs. (2.16)–(2.21) in greater detail, we compute

$$\nabla\times(\nabla\times\mathbf{E}) = \nabla\times\left(-\frac{\partial B}{\partial t}\right) = \nabla\times(i\omega\mathbf{B}) =$$
$$i\omega\left(\nabla\times\mathbf{B}\right) = i\omega\left[\mu_0\left(\mathbf{j}+\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\right] =$$
$$i\omega\mu_0\mathbf{j}+i\omega\mu_0\epsilon_0\left(-i\omega\mathbf{E}\right) = i\omega\mu_0\mathbf{j}+\omega^2\mu_0\epsilon_0\mathbf{E}$$
$$= \mu_0\omega^2\left(\frac{i}{\omega}\mathbf{j}+\epsilon_0\mathbf{E}\right) = \mu_0\omega^2\mathbf{D} = \mu_0\epsilon_0\omega^2\mathbf{KE}. \tag{2.54}$$

Now

$$\nabla\times\mathbf{E} = \left(E_{3,y}-E_{2,z}\right)\hat{\imath}+\left(E_{1,z}-E_{3,x}\right)\hat{\jmath}+\left(E_{2,x}-E_{1,y}\right)\hat{k}.$$

It is obvious that this quantity vanishes identically in the electrostatic case: apply (2.28) and equate mixed partial derivatives. But if $\nabla\times\mathbf{E}$ is itself a gradient, then the quantity $\nabla\times(\nabla\times\mathbf{E})$ vanishes for the general case as well. We will understand the seriousness of this latter difficulty once we evaluate the left-hand side of Eq. (2.54). Explicitly,

$$\nabla\times(\nabla\times\mathbf{E}) = \left(E_{2,xy}-E_{1,yy}-E_{1,zz}+E_{3,xz}\right)\hat{\imath}+$$
$$\left(E_{3,yz}-E_{2,zz}-E_{2,xx}+E_{1,yx}\right)\hat{\jmath}+\left(E_{1,zx}-E_{3,xx}-E_{3,yy}+E_{2,zy}\right)\hat{k}.$$

Applying (2.4), (2.5) to the right-hand side, we obtain the algebraic expression

$$\left[k_1k_2E_2-\left(k_2^2+k_3^2\right)E_1+k_1k_3E_3\right]\hat{\imath}+\left[k_2k_3E_3-\left(k_3^2+k_1^2\right)E_2+k_{21}E_1\right]\hat{\jmath}$$
$$+\left[k_{31}E_1-\left(k_1^2+k_2^2\right)E_3+k_{32}E_2\right]\hat{k}. \tag{2.55}$$

This object can be written as the matrix operator

$$L\mathbf{E} = \begin{pmatrix} -\left(k_2^2+k_3^2\right) & k_1k_2 & k_1k_3 \\ k_2k_1 & -\left(k_3^2+k_1^2\right) & k_2k_3 \\ k_3k_1 & k_3k_2 & -\left(k_1^2+k_2^2\right) \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}.$$

The system (2.54) is uniquely solvable if and only if the operator L can be inverted — that is, if and only if

$$\det\begin{pmatrix} -\left(k_2^2+k_3^2\right) & k_1k_2 & k_1k_3 \\ k_2k_1 & -\left(k_3^2+k_1^2\right) & k_2k_3 \\ k_3k_1 & k_3k_2 & -\left(k_1^2+k_2^2\right) \end{pmatrix} \neq 0.$$

But it is easy to check that this determinant vanishes identically for all (k_1, k_2, k_3). Of course (2.55) is just a translation of $\nabla \times (\nabla \times \mathbf{E})$ into Fourier mode. Because the symbol of a differential operator is a natural generalization of the idea of Fourier modes, we can interpret the foregoing computation to mean that the symbol of the differential operator $\nabla \times (\nabla \times \;)$ vanishes identically. This is a serious obstacle to understanding (2.54). As Weitzner notes in [40], neither the type of Eq. (2.54) (which is given by the sign of the symbol) nor the order of the equation (which is given by the degree of the symbol) are determined by standard analytic methods.

2.5.1 Choices of potential and gauge

It is therefore necessary to impose an additional hypothesis. A natural one is that an electromagnetic potential exists. But in distinction to the electrostatic case, we do not assume that $\mathbf{E}$ can be derived by simply taking the negative gradient of a scalar field.

In order to compare our computations with the extensive expositions in [39] and [40] we adopt, only in Secs. 2.5.1 and 2.5.2, the convention that the time-harmonic dependence is of the form $\exp[i\omega t]$ in units of c/ω. (This is in distinction to (2.4).) Because our time derivatives usually end up being taken twice, this only has an effect on the sign in a few intermediate calculations. However, with this sign convention, Maxwell's equations for plane waves assume the slightly different form

$$\nabla \times \mathbf{E} = -i\mathbf{B}, \tag{2.56}$$

$$\nabla \times \mathbf{B} = i\mathbf{D} = i\mathbf{K}\mathbf{E}. \tag{2.57}$$

The first choice of potentials is to let the vector $\mathbf{A}$ denote the magnetic potential and to introduce a second, scalar potential, Φ. We then write

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{2.58}$$

and

$$\mathbf{E} = -i\mathbf{A} - \nabla\Phi. \tag{2.59}$$

Taking the curl of the second equation, we obtain

$$\nabla \times \mathbf{E} = -i\nabla \times \mathbf{A} - \nabla \times \nabla\Phi. \tag{2.60}$$

Evaluating the last term on the right-hand side of (2.60) using differential forms, Φ is a zero-form and

$$\nabla \times \nabla\Phi = d^2\Phi = 0. \tag{2.61}$$

Equations (2.58) and (2.61) imply that (2.56) is satisfied under condition (2.59) for any smooth choice of $\mathbf{A}$ and Φ.

Notice that we automatically obtain from hypothesis (2.58) an extra condition

$$\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}),$$

which is to say, in terms of differential forms, that the 2-form $\mathbf{B}$ and the 1-form $\mathbf{A}$ satisfy

$$d\mathbf{B} = d^2\mathbf{A} = 0.$$

In order to evaluate (2.57), we notice, continuing to interpret the magnetic potential $\mathbf{A}$ as a 1-form and Φ as a zero-form, that if g is defined by

$$g(\mathbf{A}) = \mathbf{A} + df,$$

where f is a smooth 0-form, then

$$d(g(\mathbf{A})) = d(\mathbf{A} + df) = d\mathbf{A} + d^2 f = d\mathbf{A} = \mathbf{B},$$

so the magnetic field remains invariant under the *gauge transformation* g. Moreover, because $\delta f = 0$ for any zero-form f, we have

$$\Delta f = -(\delta d + d\delta) f = -\delta df.$$

Thus, given any smooth potential $\mathbf{A}$, we can choose f to satisfy the Poisson equation

$$\Delta f = \delta \mathbf{A},$$

in which case

$$\delta(g(\mathbf{A})) = \delta(\mathbf{A} + df) = \delta\mathbf{A} - \Delta f = 0.$$

We call the gauge produced by such a g a *Coulomb* (*transverse*, *radiation*, or *Hodge*) gauge. In vector notation,

$$i\nabla \cdot g(\mathbf{A}) = 0.$$

Computing (2.57) in the Coulomb gauge, we obtain [39]

$$\Delta g(\mathbf{A}) - i\mathbf{K}\nabla\Phi + \mathbf{K}g(\mathbf{A}) = 0.$$

Computing the symbol σ of this operator by the same method that was applied to the operator $\nabla \times (\nabla \times \mathbf{E})$, we find that $\sigma = -|\mathbf{k}|^4 (\mathbf{Kk}) \cdot \mathbf{k}$, a polynomial of degree six in $\mathbf{k}$. That the corresponding system is of order six is an expected result for a system of two first-order equations for vectors in $\mathbb{R}^3$.

Replacing the Coulomb gauge by a slightly more complicated gauge in which

$$i\nabla \cdot (\mathbf{K}^*\mathbf{A}) = 0,$$

where the superscripted asterisk denotes the adjoint matrix, we obtain a self-adjoint operator in (2.57). This more complicated gauge can be constructed by the same general method that led to the Coulomb gauge, provided that we solve a slightly more complicated Poisson problem. The symbol corresponding to this self-adjoint operator can also be calculated by the methods introduced earlier, and that symbol is also a sixth-degree polynomial in $\mathbf{k}$.

However, we can obtain a fourth-order system, which is more convenient for analysis, if we impose an additional hypothesis: that the plasma has axisymmetric geometry. While this is a very strong hypothesis, it is satisfied by plasmas produced in tokamaks.

In order to motivate the choice of potential in this case, we make a few preliminary calculations. Only for the remainder of this section, the subscripts r, θ, and z when affixed to a vector are to be interpreted as the radial, angular, and axial components of the vector unless preceded by a comma; if preceded by a comma, they are to be considered partial derivatives in the direction of the component. (The subscripted-variable notation for partial derivatives of scalar functions remains unchanged.) Adopting the basis

$$\mathbf{u}_r = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath},$$

$$\mathbf{u}_\theta = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath},$$

$$\mathbf{u}_z = -\hat{k},$$

we recall that in the axisymmetric case we can write

$$\mathbf{E} = \left(E_r\left(r,z\right)\mathbf{u}_r + E_\theta\left(r,z\right)r\mathbf{u}_\theta + E_z\left(r,z\right)\mathbf{u}_z\right)e^{im\theta},$$

and similarly for $\mathbf{B}$. If $m = 0$, the waves preserve the axisymmetry of the underlying static plasma medium, as the wave vector satisfies $\mathbf{k} = \left(k_r, 0, k_z\right).$ We will restrict our attention to this simple special case, in which

$$\begin{aligned}\nabla\times\mathbf{E} = \frac{1}{r}\left[E_{z,\theta}\mathbf{u}_r + E_{r,z}\left(r\mathbf{u}_\theta\right) + \left(rE_\theta\right)_r\mathbf{u}_z\right]\\ -\frac{1}{r}\left[E_{z,r}\left(r\mathbf{u}_\theta\right) + rE_{\theta,z}\mathbf{u}_r + E_{r,\theta}\mathbf{u}_z\right] =\\ -E_{\theta,z}\mathbf{u}_r + \left(E_{r,z} - E_{z,r}\right)\mathbf{u}_\theta + \frac{1}{r}\left(rE_\theta\right)_{,r}\mathbf{u}_z.\end{aligned} \tag{2.62}$$

Thus (2.56) implies in particular that

$$-\,iE_{\theta,z} = B_r \tag{2.63}$$

and

$$\frac{\left(rE_\theta\right)_{,r}}{r} = -iB_z. \tag{2.64}$$

Just as Eqs. (2.63) and (2.64) imply, using (2.56), that B_r and B_z can each be expressed in terms of derivatives of E_θ, so it is possible to use (2.57) to show that the other cylindrical components of $\mathbf{E}$ and $\mathbf{B}$ can be expressed as appropriate derivatives of E_θ and B_θ. This will allow the angular components of $\mathbf{E}$ and $\mathbf{B}$ to play the role of potentials for the two fields.

Applying (2.56) to the middle term of the last identity in (2.62) yields

$$E_{r,z} - E_{z,r} = -iB_\theta. \tag{2.65}$$

Because **E** and **B** have exactly analogous forms and the left-hand and middle terms of (2.57) is exactly analogous to (2.56) with a change of sign, we can immediately write

$$D_r = iB_{\theta,r}$$

and

$$D_z = -i\frac{(rB_\theta)_{,z}}{r}.$$

Now the extreme right-hand side of (2.57) yields E_r and E_z (see Eqs. (22), (23) of [39]) and one obtains

$$B_{r,z} - B_{z,r} = iD_\theta = i\left(K_{\theta r}E_r + K_{\theta\theta}E_\theta + K_{\theta z}E_z\right), \tag{2.66}$$

completing the system of equations for E_θ and B_θ.

2.5.2 Variational interpretation

Continuing to adopt the special hypotheses and special notation of Sec. 2.5.1, we continue to review the analysis in [39] of geometry-preserving plane waves in an axisymmetric plasma.

Equations (2.65), (2.66) can be associated to an energy functional:

$$\begin{aligned}\mathcal{E} = \int \{ &\left[\nabla\left(rE_\theta^*\right)\cdot\nabla\left(rE_\theta\right)\right]/r^2 + \left[\nabla\left(rB_\theta^*\right)\cdot\mathbf{K}\nabla\left(rB_\theta\right)\right]/r^2\Delta \\ &+iE_\theta\left[\left(rB_\theta^*\right)_{,r}\left(K_{zr}K_{r\theta} - K_{z\theta}K_{rr}\right)/r + B_{\theta,z}^*\left(K_{r\theta}K_{zz} - K_{rz}K_{z\theta}\right)\right]/\Delta \\ &-iE_\theta^*\left[\left(rB_\theta\right)_{,r}\left(K_{zr}K_{\theta r} - K_{\theta z}K_{rr}\right)/r + B_{\theta,z}\left(K_{\theta r}K_{zz} - K_{zr}K_{\theta z}\right)\right]/\Delta \\ &-B_\theta^*B_\theta - E_\theta^*E_\theta\left[\det\left(\mathbf{K}\right)\right]/\Delta\}r\,drdz,\end{aligned} \tag{2.67}$$

where

$$\nabla = \frac{\partial}{\partial r}r + \frac{\partial}{\partial z}z,$$

and

$$\Delta = K_{rr}K_{zz} - K_{rz}K_{zr}.$$

Provided **K** can be made self-adjoint, so can $\mathcal{E}$. Form a right-handed orthogonal set $(\mathbf{v},\theta,\mathbf{u})$, where

$$\mathbf{u} = \sin\beta r + \cos\beta k$$

and

$$\mathbf{v} = \cos\beta r - \sin\beta k.$$

The basis is to be chosen so that **u** lies in the poloidal direction and **v** lies orthogonal to it; so we write the magnetic field vector in the form

$$\mathbf{B} = B_0\left[\cos\alpha\theta + \sin\alpha\left(\sin\beta r + \cos\beta z\right)\right],$$

where α, β, and B_0 depend only on r and z. In this notation, the variational equations of $\mathcal{E}$ form a second-order system in which the differential operator for

one of the equations is essentially the Laplacian $\mathcal{L}$. We ignore that equation, as standard analytic methods can be applied to it. The differential operator for the other equation looks like

$$\mathcal{L} + (\mathbf{u} \cdot \nabla)^2 , \tag{2.68}$$

and that is the equation — in particular, the second of the two differential operators in that equation — that we will study in the remainder of this review. The term (2.68) in Eq. (2.67) can be written explicitly, in terms of the chosen basis, in the form

$$\begin{aligned} r\nabla \cdot \left[\left(\frac{\xi}{r^2 \Delta} \right) \nabla (rB_\theta) \right] - r\nabla \cdot \left[\left(\frac{\zeta \sin^2 \alpha}{r^2 \Delta} \right) (\mathbf{u} \cdot \nabla) (rB_\theta) \mathbf{u} \right] \\ -i\theta \cdot \nabla (rB_\theta) \times \nabla \left(\frac{\mu \cos \alpha}{r\Delta} \right) + B_\theta = \\ (r\Delta)^{-1} \left[\mu (\zeta - \xi) \sin \alpha \mathbf{u} \cdot \nabla (rE_\theta) + i \left(\mu^2 - \xi\zeta \right) \sin \alpha \cos \alpha \mathbf{v} \cdot \nabla (rE_\theta) \right] , \end{aligned} \tag{2.69}$$

where

$$\xi = 1 + \sum_{\nu=1}^{N} \frac{\Pi_\nu^2}{\Omega_\nu^2 - \omega^2},$$

$$\zeta = \xi + \sum_{\nu=1}^{N} \frac{\Pi_\nu^2}{\omega^2} - 1,$$

and

$$\mu = \sum_{\nu=1}^{N} \frac{\Pi_\nu^2 \Omega_\nu}{\omega \left(\Omega_\nu^2 - \omega^2 \right)}.$$

Equation (2.69) is only elliptic for negative values of ξ. Physically, this is the condition for so-called *lower-hybrid* frequencies, at which

$$1 + \frac{\Pi_e^2}{\Omega_e^2} < \frac{\Pi_i^2}{\omega^2},$$

c.f. (2.37). Noticing that ξ is a function of r and z, define a new variable $\eta(r, z)$ so the curves $\eta =$ constant are orthogonal to the curves $\xi =$ constant. Rewriting (2.69) in (ξ, η)-coordinates, the behavior of the solution depends on whether or not

$$\mathbf{u} \cdot \nabla \xi = 0.$$

This identity implies that flux surfaces coincide with resonance surfaces. In that case, Eq. (2.69) is analogous to Eq. (2.46) of Sec. 2.4.3 and is not of Tricomi type. The second-order terms of that equation can be written in the form

$$L(u) = f(\xi, \eta) \left[\xi u_{\xi\xi} + M(\xi, \eta) u_{\eta\eta} \right] , \tag{2.70}$$

where $u = u(\xi, \eta)$ is a scalar function; f and M are given well behaved scalar functions near $\xi = 0$ and, in addition, M is positive.

The physical model allows two further alternatives: If the curve representing the flux surface in two dimensions is collinear with the resonance curve as in (2.70), then the plasma can be treated as a perpendicularly stratified medium, which is essentially the case considered in Sec. 2.4.1. If the resonance curve is tangent to the curve representing the flux surface, then we are in a more mathematically and physically interesting case. In this latter case, the simplest model for the operator L of (2.69) is an operator for which the highest-order terms have the form

$$\tilde{L}(u) = \left(x - y^2\right) u_{xx} + u_{yy}. \tag{2.71}$$

Note that this operator is closely related to the differential operator of Eq. (2.53). The two operators can be made virtually identical by replacing the coordinate transformation (2.51) by

$$x \to \tilde{x} = -x/a. \tag{2.72}$$

2.6 A conjecture about the singular set

Methods for deriving the smoothness of solutions to the Tricomi equation appear to fail for an operator of the form (2.71) whenever the domain includes the origin of coordinates. This suggests the existence of a singular point at the origin, a conjecture which is supported by an analysis of characteristic lines.

In order for a characteristic line to pass through the origin, the point(x, y) would need to satisfy the identity

$$x = \lambda y^2 \tag{2.73}$$

for some constant λ, and also the characteristic equation for (2.71). Substituting (2.73) into the characteristic equation

$$\left(x - y^2\right) dy^2 + dx^2 = 0, \tag{2.74}$$

one obtains the equation

$$\frac{dy^2}{\left(2\lambda y dy\right)^2} = \frac{1}{\left(1 - \lambda\right) y^2}, \tag{2.75}$$

or

$$4\lambda^2 + \lambda - 1 = 0.$$

This polynomial has two real solutions; considering that the characteristic equation (2.75) has two roots, one concludes [26] that four characteristic lines must pass through the origin — two more than pass through any other hyperbolic point. This motivates the suspicion that solutions of at least the equation $\tilde{L}u = 0$ will tend to be singular at the origin. It has been observed that an energy sink or plasma heating zone might be associated with such a singularity; see [16], [26], [33], [39], and [40] for details on this and other issues raised in this section.

3 Analysis of the model equation

Physical reasoning suggests that the *closed* Dirichlet problem, in which data are prescribed along the entire boundary of the domain, should be well-posed for the cold plasma model. However, the closed Dirichlet problem has been shown to be ill-posed, in the classical sense, for the equation

$$\left(x - y^2\right) u_{xx} + u_{yy} + \frac{1}{2} u_x = 0$$

on a typical domain [26]. This leads us to ask whether a well-posed problem with closed boundary data can be formulated in a suitably weak sense. In this section we address the "existence" part of that question.

Because the operator introduced in Eq. (2.71) is not of Tricomi type at the origin, where it satisfies a condition of the form (2.42), we expect weaker regularity than we have for operators which are uniformly of Tricomi type. In particular, although we can show the existence of very weak solutions in L^2, we do not expect H^1 regularity for the closed Dirichlet problem. This lack of optimism is supported by numerical experiments [26]. Moreover, our methods are insufficient to determine the uniqueness of a solution.

Denote by Ω a bounded, connected domain of $\mathbb{R}^2$ having piecewise smooth boundary $\partial\Omega$, oriented in a counterclockwise direction; the domain includes both an arc of the sonic curve and the origin of coordinates in $\mathbb{R}^2$. (This insures that our equation will be elliptic-hyperbolic but not equivalent to an equation of Tricomi type.)

Define [21], for a given C^1 function $\mathcal{K}\left(x, y\right),$ the space $L^2\left(\Omega; |\mathcal{K}|\right)$ and its dual. These spaces consist, respectively, of functions u for which the norm

$$||u||_{L^2(\Omega;|\mathcal{K}|)} = \left(\int\int_\Omega |\mathcal{K}| u^2 dxdy\right)^{1/2}$$

is finite, and functions $u \in L^2\left(\Omega\right)$ for which the norm

$$||u||_{L^2(\Omega;|\mathcal{K}|^{-1})} = \left(\int\int_\Omega |\mathcal{K}|^{-1} u^2 dxdy\right)^{1/2}$$

is finite. Standard arguments allow us to define the space $H_0^1(\Omega; \mathcal{K})$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$||u||_{H^1(\Omega;\mathcal{K})} = \left[\int\int_\Omega \left(|\mathcal{K}| u_x^2 + u_y^2 + u^2\right) \, dxdy\right]^{1/2}. \tag{3.1}$$

The $H_0^1(\Omega; \mathcal{K})$-norm has the explicit form

$$||u||_{H_0^1(\Omega;\mathcal{K})} = \left[\int\int_\Omega \left(|\mathcal{K}| u_x^2 + u_y^2\right) \, dxdy\right]^{1/2}, \tag{3.2}$$

which can be derived from (3.1) via a weighted Poincaré inequality.

In the following we denote by C generic positive constants, the value of which may change from line to line.

3.1 The closed Dirichlet problem for distribution solutions

Consider the equation

$$Lu = f, \tag{3.3}$$

where f is a given, sufficiently smooth function of (x, y) and

$$L = \left(x - y^2\right)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \kappa\frac{\partial}{\partial x} \tag{3.4}$$

for a given constant κ. By a *distribution solution* of equations (3.3), (3.4) with the boundary condition

$$u(x, y) = 0 \, \forall (x, y) \in \partial\Omega \tag{3.5}$$

we mean a function $u \in L^2(\Omega)$ such that $\forall \xi \in H_0^1(\Omega; \mathcal{K})$ for which $L^*\xi \in L^2(\Omega)$, we have

$$(u, L^*\xi) = \langle f, \xi \rangle. \tag{3.6}$$

Here L^* is the adjoint operator; $(\ ,\ )$ denotes the L^2 inner product on Ω; $\langle\ ,\ \rangle$ is the *duality bracket* associated to the H^{-1} norm

$$||w||_{H^{-1}(\Omega;\mathcal{K})} = \sup_{0 \neq \xi \in C_0^\infty(\Omega)} \frac{|\langle w, \xi \rangle|}{||\xi||_{H_0^1(\Omega;\mathcal{K})}}.$$

Such solutions have also been called *weak; c.f.* Eq. (2.13) of [4], Sec. II.2. In fact they are a little smoother than generic distribution solutions, as they lie in a classical function space.

The existence of solutions to boundary value problems can be shown to follow from *energy inequalities* having the general form

$$||v||_V \leq C||L^*v||_U,$$

where U and V are suitable function spaces. We will combine such an inequality with the Riesz Representation Theorem to prove the existence of distribution solutions; see [4] for a general reference.

Lemma 3.1. ([28]). *The inequality*

$$||u||_{H_0^1(\Omega;\mathcal{K})} \leq C||Lu||_{L^2(\Omega)},$$

is satisfied for $u \in C_0^2(\Omega)$, *where the positive constant* C *depends on* Ω *and* $\mathcal{K}$; L *is defined by (3.4) with* $\kappa \in [0, 2]$; $\mathcal{K} = x - y^2$.

Proof. We outline the proof; for details, see [28], Sec. 2. Initially, let $1 \leq \kappa \leq 2$, and let δ be a small, positive constant. Define an operator M by the identity

$$Mu = au + bu_x + cu_y$$

for $a=-1$, $c=2\left(2\delta-1\right)y$, and

$$b=\left\{\begin{array}{ll}\exp\left(2\delta\mathcal{K}/Q_1\right) & \text{if } (x,y)\in\Omega^+\\ \exp\left(6\delta\mathcal{K}/Q_2\right) & \text{if } (x,y)\in\Omega^-\end{array}\right.,$$

where $\Omega^+=\{(x,y)\in\Omega\,|\,\mathcal{K}>0\}$ and $\Omega^-=\Omega\backslash\Omega^+$. Choose $Q_1=\exp\left(2\delta\mu_1\right)$, where $\mu_1=\max_{(x,y)\in\overline{\Omega^+}}\mathcal{K}$. Define the negative number $\mu_2=\min_{(x,y)\in\overline{\Omega^-}}\mathcal{K}$ and let $Q_2=\exp\left(\mu_2\right)$. Notice that $b\leq Q_1$ on Ω^+ and $b>Q_2$ on Ω^-.

We will estimate the quantity (Mu,Lu) from above and below. As in the Tricomi case [21], one of the coefficients in Mu fails to be continuously differentiable on all of Ω. When integrating this quantity, a cut should be introduced along the line $\mathcal{K}=0$. The boundary integrals involving a, b, and c on either side of this line will cancel.

The boundary terms vanish by the compact support of u. Integration by parts yields the identity

$$(Mu,Lu)=\int\int_{\Omega^+\cup\Omega^-}\alpha u_x^2+2\beta u_xu_y+\gamma u_y^2\,dxdy,$$

where

$$\alpha=\left(\frac{c_y}{2}-a-\frac{b_x}{2}\right)\mathcal{K}+\left(\kappa-\frac{1}{2}\right)b-cy,$$

for

$$\alpha_{|\Omega^+}=\left(2-\frac{b}{Q_1}\right)\delta\mathcal{K}+2\left(1-2\delta\right)y^2+\left(\kappa-\frac{1}{2}\right)b$$

and

$$\alpha_{|\Omega^-}=\left(3\frac{b}{Q_2}-2\right)\delta|\mathcal{K}|+2\left(1-2\delta\right)y^2+\left(\kappa-\frac{1}{2}\right)b;$$

$$\beta=\frac{1}{2}\left[c\left(\kappa-1\right)-b_y\right]=\left\{\begin{array}{ll}y\left[2\delta\left(b/Q_1\right)+\left(\kappa-1\right)\left(2\delta-1\right)\right]\leq|y| & \text{in } \Omega^+\\ y\left[6\delta\left(b/Q_2\right)+\left(\kappa-1\right)\left(2\delta-1\right)\right]\leq\kappa|y| & \text{in } \Omega^-\end{array}\right.;$$

$$\gamma=\frac{1}{2}\left(b_x-c_y\right)-a=\left\{\begin{array}{ll}2\left(1-\delta\right)+\delta\left(b/Q_1\right) & \text{in } \Omega^+\\ 2\left(1-\delta\right)+3\delta\left(b/Q_2\right) & \text{in } \Omega^-\end{array}\right..$$

On Ω^+, for any scalars ξ and η, we have

$$\alpha\xi^2+2\beta\xi\eta+\gamma\eta^2\geq\alpha\xi^2-\left(y^2\xi^2+\eta^2\right)+\gamma\eta^2=$$

$$\left[\left(2-\frac{b}{Q_1}\right)\delta\mathcal{K}+\left(1-4\delta\right)y^2+\left(\kappa-\frac{1}{2}\right)b\right]\xi^2+\left[\left(1-2\delta\right)+\frac{6b}{Q_1}\right]\eta^2$$

$$\geq\delta\left(\mathcal{K}\xi^2+\eta^2\right),$$

provided δ is sufficiently small. On Ω^-,

$$\alpha\xi^2+2\beta\xi\eta+\gamma\eta^2\geq\alpha^2\xi^2-2\left(y^2\xi^2+\eta^2\right)+\gamma\eta^2=$$

$$\left[\left(3\frac{b}{Q_2}-2\right)\delta|\mathcal{K}|-4\delta y^2+\left(\kappa-\frac{1}{2}\right)b\right]\xi^2+\delta\left(3\frac{b}{Q_2}-2\right)\eta^2$$

$$\geq \delta\left(|\mathcal{K}|\xi^2+\eta^2\right).$$

Arguing in this way on each subdomain (and taking $\xi = u_x$, $\eta = u_y$), we obtain

$$(Mu, Lu) \geq \delta ||u||^2_{H^1_0(\Omega;\mathcal{K})}. \tag{3.7}$$

For the upper estimate, we have [21]

$$(Mu, Lu) \leq ||Mu||_{L^2}||Lu||_{L^2} \leq C\,(K,\Omega)\,||u||_{H^1_0(\Omega;\mathcal{K})}||Lu||_{L^2(\Omega)}. \tag{3.8}$$

Combining (3.7) and (3.8), and dividing through by the weighted H^1_0-norm of u, completes the proof for the case $\kappa \in [1,2]$.

Now let $0 \leq \kappa < 1$. Again subdivide the domain into Ω^+ and Ω^- by introducing a cut along the curve $K = 0$. Integrate by parts, choosing $a = -1$;

$$b = \begin{cases} -N\mathcal{K} & \text{in } \Omega^+ \\ N\mathcal{K} & \text{in } \Omega^- \end{cases},$$

where N is a constant satisfying

$$\frac{1+\tilde{\delta}}{3-\kappa} < N < \frac{1-\tilde{\delta}}{\kappa+1} \tag{3.9}$$

for a sufficiently small positive constant $\tilde{\delta}$; $c = -4Ny$. The boundary integrals involving a and c on either side of the cut will cancel and the boundary integrals involving b will be zero along the cut. Inequality (3.7) can be derived by an argument broadly analogous to the case $\kappa \in [1,2]$. Applying (3.8) then completes the proof. □

Theorem 3.1 ([28]). *The Dirichlet problem (3.3), (3.4), (3.5) with $\kappa \in [0,2]$ possesses a distribution solution $u \in L^2(\Omega)$ for every $f \in H^{-1}(\Omega;\mathcal{K})$.*

Proof. Again, we only outline the proof (*c.f.* [21], Theorem 2.2). Define for $\xi \in C^\infty_0$ a linear functional

$$J_f(L\xi) = \langle f, \xi\rangle.$$

This functional is bounded on a subspace of L^2 by the inequality

$$|\langle f,\xi\rangle| \leq ||f||_{H^{-1}(\Omega;\mathcal{K})}||\xi||_{H^1_0(\Omega;\mathcal{K})} \tag{3.10}$$

and by applying Lemma 3.1 to the second term on the right. Precisely, J_f is a bounded linear functional on the subspace of $L^2(\Omega)$ consisting of elements having the form $L\xi$ with $\xi \in C^\infty_0(\Omega)$. Extending J_f to the closure of this subspace by Hahn-Banach arguments, we obtain a functional defined on all of L^2. The Riesz Representation Theorem then guarantees the existence of a distribution solution in the self-adjoint case. If $\kappa \neq 1$, then L is not self-adjoint, and

$$L^* = \left(x-y^2\right)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (2-\kappa)\frac{\partial}{\partial x}. \tag{3.11}$$

Estimating L for κ in $[0,2]$ will also yield estimates for the adjoint L^*. Applying the preceding argument to the adjoint operator completes the proof of Theorem 3.1. □

3.2 Mixed boundary value problems with closed boundary data

It is also possible to form *mixed Dirichlet-Neumann* problems for operators of the form (3.4). Mixed boundary value problems arise in various contexts in plasma physics (*e.g.*, [7]) and in related topics from electromagnetic theory (*e.g.*, [19], which is related to the model of Sec. 2.4.1). However, the results of this section also imply — by taking the set of boundary points on which the Dirichlet conditions are imposed to be empty — the existence of weak solutions to a class of Neumann problems.

Denote by $\mathbf{u} = (u_1, u_2)$ and $\mathbf{w} = (w_1, w_2)$ measurable vector-valued functions on Ω. Define $\mathcal{H}_{\mathfrak{K}}$ to be the Hilbert space of measurable functions on Ω for which the norm induced in the obvious way by the weighted L^2 inner product

$$(\mathbf{u}, \mathbf{w})_{\mathfrak{K}} = \int\int_{\Omega} (|\mathcal{K}| u_1 w_1 + u_2 w_2)\, dxdy$$

is finite. In the notation for these spaces, $\mathfrak{K}$ denotes a diagonal matrix having entries $|\mathcal{K}|$ and 1.

By a *weak solution* of a mixed boundary-value problem in this context we mean an element $\mathbf{u} \in \mathcal{H}_{\mathfrak{K}}(\Omega)$ such that

$$-\left(\mathbf{u}, \mathcal{L}^* \mathbf{w}\right)_{L^2(\Omega;\mathbb{R}^2)} = (\mathbf{f}, \mathbf{w})_{L^2(\Omega;\mathbb{R}^2)} \tag{3.12}$$

for every function $\mathbf{w} \in C^1\left(\overline{\Omega}; \mathbb{R}^2\right)$ for which $\mathfrak{K}^{-1}\mathcal{L}^*\mathbf{w} \in L^2\left(\Omega; \mathbb{R}^2\right)$ and for which

$$w_1 = 0 \,\forall\, (x, y) \in G \tag{3.13}$$

and

$$w_2 = 0 \,\forall\, (x, y) \in \partial\Omega \backslash G, \tag{3.14}$$

where $G \subset \partial\Omega$. Choose the differential operator $\mathcal{L}$ to have the form

$$\begin{pmatrix} \mathcal{K}\partial_x & \partial_y \\ \partial_y & -\partial_x \end{pmatrix} + \begin{pmatrix} \kappa & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.15}$$

where κ is a number in $[0, 1]$.

Theorem 3.2 ([28]). *Let G be a subset of $\partial\Omega$ and let $\mathcal{K} = x - y^2$. Define the functions $b(x, y) = m\mathcal{K} + s$ and $c(y) = \mu y - t$, where μ is a positive constant,*

$$m = \begin{cases} (\mu + \delta)/2 & in\ \Omega^+ \\ (\mu - \delta)/2 & in\ \Omega^- \end{cases}$$

for a small positive number δ, and t is a positive constant such that $\mu y - t < 0\, \forall y \in \Omega$. Let s be a sufficiently large positive constant. In particular, choose s to be so large that the quantities $m\mathcal{K} + s$, $2cy + s$, and $b^2 + \mathcal{K}c^2$ are all positive. Let

$$bdy - cdx \le 0 \tag{3.16}$$

on G and

$$\mathcal{K}\,(bdy - cdx) \geq 0 \tag{3.17}$$

on $\partial\Omega\backslash G$. Then there exists for every $\mathbf{f}$ *such that* $\mathfrak{K}^{-1}\mathcal{M}^T\mathbf{f} \in L^2(\Omega)$ *a weak solution to the mixed boundary-value problem (3.12)–(3.14) for* $\mathcal{L}$ *given by Eq. (3.15) with* $\kappa = 0$*, where the superscripted* T *denotes matrix transpose.*

Proof. We give the idea of the proof [28]. One shows that there exists a positive constant C such that

$$(\mathbf{\Psi}, \mathcal{L}^*\mathcal{M}\mathbf{\Psi}) \geq C \int\int_\Omega \left(|\mathcal{K}|\Psi_1^2 + \Psi_2^2\right) dxdy$$

for any sufficiently smooth 2-vector $\mathbf{\Psi}$, provided conditions (3.13), (3.14) are satisfied on the boundary for $\mathbf{w} = \mathcal{M}\mathbf{\Psi}$, where $\mathcal{L}^*$ is given by (3.15) with $\kappa = 1$ and

$$\mathcal{M} = \begin{pmatrix} b & c \\ -\mathcal{K}c & b \end{pmatrix}.$$

This inequality leads to an application of the Riesz Representation Theorem by arguments which are roughly analogous to those used to prove Theorem 3.1. □

Despite the technical nature of the hypotheses in Theorem 3.2, simple domains which satisfy them are very easy to construct — *e.g.*, a box in the first quadrant having a vertex at the origin of coordinates, or a narrow lens about the sonic curve in the first quadrant. Note that by taking G to be the empty set, we obtain a solution to the closed conormal problem (*c.f.* [32]). But in order for Theorem 3.2 to guarantee a solution to the closed Dirichlet problem, we would need to find a domain on which G could be taken to be the entire boundary; it is not obvious how to construct such a domain. And, as was also the case in Theorem 3.1, the proof of Theorem 3.2 does not establish the uniqueness of solutions.

In addition to its intrinsic mathematical and physical interest, the formulation of boundary value problems illuminates other topics in the analysis of the cold plasma model. For example, it is shown in Sec. 2.4.3, by a tedious analytic argument, that away from the origin the governing equation for the model is of Tricomi type, whereas in the neighborhood of the origin it is of Keldysh type. This distinction is also suggested, without reference to such terminology, by other analytic arguments in [33] and in Sec. 4 of [40]. If we try to form a standard elliptic-hyperbolic boundary value problem in which the hyperbolic region is composed of intersecting characteristics, we might choose both these characteristics to originate at points on the arc of the resonance curve $x = y^2$ that lies in the first quadrant, or both of them to lie in the fourth quadrant. We then obtain a standard problem for a vertical-ice-cream-cone-shaped region (in the former case, the ice-cream cone is held upside down), similar to those formulated for the Tricomi equation (Eq. (2.44) with $\mathcal{K}\,(\xi, \eta) = \xi$). The domain geometry is exactly analogous to, for example, Fig. 2 of [25], with the line AB in that figure replaced by an arc of the curve $x = y^2$, lying either completely

above or completely below the x-axis. But the origin will not be included, as that is a singular point of the characteristic equation (2.74). If we include the origin, we are led to a hyperbolic region bounded by characteristics in the second and third quadrants, a horizontal-ice-cream-cone-shaped region similar to those formulated for the Cinquini-Cibrario equation (Eq. (2.46) with $\mathcal{K}(\xi,\eta)=\xi$). In this case typical domain geometry is analogous to Fig. 2 of [10], with the line MN in that figure replaced by an arc of the curve $x=y^2$ which is symmetric about the x-axis; see also Remark *i)* following Corollary 11 of [28]. Thus the defining analytic character of the equation is clearly apparent in the geometry of the natural boundary value problems.

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